Econ 103: Topics in Single Linear Regression

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Advanced Inference Topics

Evaluating our Model

Modeling Choices

Inference: Linear Combinations of Parameters

Recall that, approximately for large n

$$\sqrt{n}(\hat{\beta}_0 - \beta_0) \sim N\left(0, \mathbb{E}[X^2]\sigma_\epsilon^2/\sigma_X^2\right), \quad \sqrt{n}(\hat{\beta}_1 - \beta_1) \sim N\left(0, \sigma_\epsilon^2/\sigma_X^2\right)$$

and $\sigma_{\beta_{01}} = \text{Cov}(\sqrt{n}\{\hat{\beta}_0 - \beta_0\}, \sqrt{n}\{\hat{\beta}_1 - \beta_1\}) = -\mathbb{E}[X]\frac{\sigma_\epsilon^2}{\sigma_X^2}$.

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These results were also often presented in the following equivalent manners

$$\frac{\hat{\beta}_0 - \beta_0}{\sigma_{\beta_0}/\sqrt{n}} \sim N(0,1) \quad \text{and} \quad \hat{\beta}_0 \sim N(\beta_1, \sigma_{\beta_0}^2/n)$$

$$\frac{\hat{\beta}_1 - \beta_1}{\sigma_{\beta_1}/\sqrt{n}} \sim N(0,1) \quad \text{and} \quad \hat{\beta}_1 \sim N(\beta_1, \sigma_{\beta_1}^2/n)$$

where $\sigma_{\beta_0}^2 = \mathbb{E}[X^2]\sigma_{\epsilon^2}/\sigma_X^2$ and $\sigma_{\beta_1}^2 = \sigma_\epsilon^2/\sigma_X^2$.

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- Also went over how to estimate these variances

Inference: Linear Combinations of Parameters

In the last lecture, we used these distributional results to compute objects like

$$\Pr\left(|\hat{\beta}_1| > 5 \mid \beta_1 = 0\right).$$

which in turn were useful for hypothesis testing

$$H_0 : \beta_1 = 0 \quad \text{vs.} \quad H_1 : \beta_1 \neq 0.$$

However, often we want to preform inference not just on one parameter, but on a linear combination of parameters, i.e we want to test

$$H_0 : \beta_0 + 5\beta_1 = 0 \quad \text{vs.} \quad H_1 : \beta_0 + 5\beta_1 \neq 0.$$

However, often we want to preform inference not just on one parameter, but on a linear combination of parameters, i.e we want to test

$$H_0 : \beta_0 + 5\beta_1 = 0 \quad \text{vs.} \quad H_1 : \beta_0 + 5\beta_1 \neq 0.$$

This is useful, for example, if we are trying to test something like

$$H_0 : \mathbb{E}[Y|X = 5] = 0 \quad \text{vs.} \quad H_1 : \mathbb{E}[Y|X = 5] \neq 0$$

and we view the linear regression model $Y = \beta_0 + \beta_1 X + \epsilon$ as a way of approximating the conditional mean function $\mathbb{E}[Y|X = x]$.

In order to test such a hypothesis we want to know the distribution of a linear combination of our model parameters. That is, for $\lambda = a\beta_0 + b\beta_1$ we would like to know the approximate distribution of

$$\hat{\lambda} = a\hat{\beta}_0 + b\hat{\beta}_1$$

so that we can calculate objects like $\Pr(|\hat{\lambda}| > 0.5 \mid \lambda = 0)$.

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Note that

$$\sqrt{n}\left(\hat{\lambda} - \lambda\right) = \sqrt{n}\left(a\hat{\beta}_0 + b\hat{\beta}_1 - a\beta_0 - b\beta_1\right)$$
$$= a\sqrt{n}\left(\hat{\beta}_0 - \beta_0\right) + b\sqrt{n}\left(\hat{\beta}_1 - \beta_1\right).$$

and that we know the (joint) distribution of $\sqrt{n}(\hat{\beta}_0 - \beta_0)$ and $\sqrt{n}(\hat{\beta}_1 - \beta_1)$.

Recall from our Econ 41 Review that the sum of two jointly normal random variables is also normally distributed and that if X and Y are random variables then

$$\mathrm{Var}(aX + bY) = a^2 \mathrm{Var}(X) + b^2 \mathrm{Var}(Y) + 2ab \mathrm{Cov}(X, Y).$$

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$$\text{Var}(aX + bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y) + 2ab \text{Cov}(X, Y).$$

Using this result along with $X = \sqrt{n}(\hat{\beta}_0 - \beta_0)$ and $Y = \sqrt{n}(\hat{\beta}_1 - \beta_1)$ gives us that, for large n:

$$\sqrt{n}\left(\hat{\lambda} - \lambda\right) \sim N(0, \sigma_\lambda^2) \implies \frac{\hat{\lambda} - \lambda}{\sigma_\lambda/\sqrt{n}} \sim N(0, 1),$$

where $\sigma_\lambda^2 = a^2 \sigma_{\beta_0}^2 + b^2 \sigma_{\beta_1}^2 + 2ab\sigma_{\beta_{01}}$

As a reminder, we can estimate

$$\sigma_{\beta_0}^2 = \mathbb{E}[X^2]\frac{\sigma_\epsilon^2}{\sigma_X^2} \iff \hat{\sigma}_{\beta_0}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \cdot \frac{\hat{\sigma}_\epsilon^2}{\hat{\sigma}_X^2}$$

$$\sigma_{\beta_1}^2 = \frac{\sigma_\epsilon^2}{\sigma_X^2} \iff \hat{\sigma}_{\beta_1}^2 = \frac{\hat{\sigma}_\epsilon^2}{\hat{\sigma}_X^2}$$

$$\sigma_{\beta_{01}} = -\mathbb{E}[X]\frac{\sigma_\epsilon^2}{\sigma_X^2} \iff \hat{\sigma}_{\beta_{01}} = \bar{X}\frac{\hat{\sigma}_\epsilon^2}{\hat{\sigma}_X^2}$$

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So, we can use these to estimate $\sigma_\lambda^2 = a^2\sigma_{\beta_0}^2 + b^2\sigma_{\beta_1}^2 + 2ab\sigma_{\beta_{01}}$ with

$$\hat{\sigma}_\lambda^2 = a^2\hat{\sigma}_{\beta_0}^2 + b^2\hat{\sigma}_{\beta_1}^2 + 2ab\hat{\sigma}_{\beta_{01}}.$$

As $n \to \infty$, $\hat{\sigma}^2_{\beta_0} \to \sigma^2_{\beta_0}$, $\hat{\sigma}^2_{\beta_1} \to \sigma^2_{\beta_1}$, and $\hat{\sigma}_{\beta_{01}} \to \sigma_{\beta_{01}}$ by the Law of Large Numbers. This gives us that $\hat{\sigma}^2_\lambda \to \sigma^2_\lambda$ as $n \to \infty$ so that we can say (approximately for large n):

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$$\frac{\hat{\lambda} - \lambda}{\hat{\sigma}_{\lambda}/\sqrt{n}} \sim N(0, 1).$$

As when considering just $\hat{\beta}_0$ or $\hat{\beta}_1$, this distributional result will be useful for hypothesis testing and creating confidence intervals.

Using the distributional result:

$$\frac{\hat{\lambda} - \lambda}{\hat{\sigma}_\lambda / \sqrt{n}} \sim N(0, 1),$$

we can test a null hypothesis of the form $H_0 : \lambda \leq \ell$, $H_0 : \lambda \geq \ell$, or $H_0 : \lambda = \ell$ by first constructing our test statistic

$$t^* = \frac{\hat{\lambda} - \ell}{\hat{\sigma}_\lambda / \sqrt{n}}.$$

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As before, we want to reject our null hypothesis if the probability of obtaining our test statistic (or something even further from the null hypothesis) under the null hypothesis is less than or equal to some pre-specified value α.

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- Recall that by the distributional result, under the null $t^* \sim N(0, 1)$

- The quantity $\hat{\sigma}_\lambda / \sqrt{n}$ is called the standard error of $\hat{\lambda}$.

Now that we have constructed our test statistic t^* we can conduct our test in two (equivalent) ways, as before

1. Construct a p-value and reject if $p < \alpha$:

   ○ If $H_0 : \lambda \leq \ell$ and $H_1 : \lambda > \ell$:

   $$p = \Pr(Z \geq t^*).$$

   ○ If $H_0 : \lambda \geq \ell$ and $H_1 : \lambda < \ell$:

   $$p = \Pr(Z \leq t^*).$$

   ○ If $H_0 : \lambda = \ell$ and $H_1 : \lambda \neq \ell$:

   $$p = \Pr(|Z| \geq |t^*|) = 2\Pr(Z \geq |t^*|).$$

Inference: Linear Combinations of Parameters

Now that we have constructed our test statistic t^* we can conduct our test in two (equivalent) ways, as before

2. Compare the t statistic to the $1 - \alpha$ or $1 - \alpha/2$ quantile of the standard normal distribution: $z_{1-\alpha}$ or $z_{1-\alpha/2}$.

   ○ If $H_0 : \lambda \leq \ell$ and $H_1 : \lambda > \ell$ reject if

   $$t^* \geq z_{1-\alpha}.$$

   ○ If $H_0 : \lambda \geq \ell$ and $H_1 : \lambda < \ell$ reject if

   $$t^* \leq -z_{1-\alpha}.$$

   ○ If $H_0 : \lambda = \ell$ and $H_1 : \lambda \neq \ell$ reject if

   $$|t^*| \geq z_{1-\alpha/2}.$$

   As a reminder $z_{1-\alpha}$ and $z_{1-\alpha/2}$ are such that

   $$\Pr(Z \leq z_{1-\alpha}) = 1 - \alpha \iff \Pr(Z > z_{1-\alpha}) = \alpha$$
   $$\Pr(Z \leq z_{1-\alpha/2}) = 1 - \alpha/2 \iff \Pr(|Z| > z_{1-\alpha/2}) = \alpha$$

We can also construct a $100(1-\alpha)\%$ confidence interval for λ in the same way as before: by looking at the values of ℓ for which we would fail to reject the null hypothesis $H_0 : \lambda = \ell$ against a two-sided alternative $H_1 : \lambda \neq \ell$ at level α.

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This gives us a symmetric formula as before, a $100(1-\alpha)\%$ confidence interval for λ is given

$$\hat{\lambda} \pm z_{1-\alpha/2}\frac{\hat{\sigma}_\lambda}{\sqrt{n}}.$$

As an aside, we can start to see a pattern here. Essentially anytime we have a distributional result like

$$\frac{\text{Estimator} - \text{True Value}}{\text{Standard Error of Estimator}} \sim N(0,1).$$

we can test a null hypothesis by constructing our test statistic

$$t^* = \frac{\text{Estimator} - \text{Null Hypothesis Value}}{\text{Standard Error of Estimate}}.$$

and then computing a p-value or directly compating this test statistic to $z_{1-\alpha}$, $-z_{1-\alpha}$, or $z_{1-\alpha/2}$ (depending on what alternate hypothesis we are testing).

We can also use this distributional result to generate $100(1 - \alpha)\%$ confidence intervals for the true value via

$$\text{Estimator} \pm z_{1-\alpha/2} \cdot \text{Standard Error of Estimator}.$$

Questions?

Example: Suppose we are arguing with our professional colleague Kyle Kuzma about the relationship between number of mental health days taken in a month (X) and the average number of points per game scored in the NBA (Y). Kuzma claims that $\mathbb{E}[Y|X=3] = 20$, we want to test this claim at level $\alpha = 0.05$.

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First we collect a random sample of 49 NBA players and ask them how many mental health days they took this month and their average points per game, $\{Y_i, X_i\}_{i=1}^{49}$. Then, since we believe the relationship between Y and X to be linear, we estimate the linear model

$$Y = \beta_0 + \beta_1 \cdot X + \epsilon.$$

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$$Y = \beta_0 + \beta_1 \cdot X + \epsilon.$$

We can then estimate $\mathbb{E}[Y|X = 3]$ by $\hat{\beta}_0 + 3\hat{\beta}_1$.

Inference: Linear Combinations of Parameters

We can test Kuzma's claim that $\mathbb{E}[Y|X = 3] = 20$ by running the following hypothesis test

$$H_0 : \beta_0 + 3\beta_1 = 20 \quad \text{vs.} \quad H_1 : \beta_0 + 3\beta_1 \neq 20.$$

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To test this claim we use our data (with $n = 49$) to estimate

$$\hat{\beta}_0 = 10, \quad \hat{\beta}_1 = 3$$

$$\hat{\sigma}^2_{\beta_0} = \hat{\sigma}^2_{\beta_1} = \hat{\sigma}_{\beta_{01}} = 1$$

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Using these estimates we get

$$\hat{\lambda} = \hat{\beta}_0 + 3\hat{\beta}_1 = 19$$
$$\hat{\sigma}^2_{\lambda} = \hat{\sigma}^2_{\beta_0} + 9\hat{\sigma}^2_{\beta_1} + 6\hat{\sigma}_{\beta_{01}} = 16$$

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- Notice how much larger $\hat{\sigma}^2_{\lambda}$ is than $\hat{\sigma}^2_{\beta_0}$ or $\hat{\sigma}^2_{\beta_1}$.

Using $\hat{\lambda} = 19$, $\hat{\sigma}_\lambda^2 = 16$, and $n = 49$ we can construct our test statistic for
$H_0 : \lambda = 20$ vs $H_1 : \lambda \neq 20$

$$t^* = \frac{\hat{\lambda} - 20}{\hat{\sigma}_\lambda/\sqrt{n}} = \frac{19 - 20}{\sqrt{16}/\sqrt{49}} = -\frac{1}{4/7} = -\frac{7}{4} = -1.75.$$

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We'll run our test in two ways. First, let's compute our p-value

$$p = \Pr(|Z| \geq |-1.75|) = 2\Pr(Z \geq 1.75) = 2(1 - \Pr(Z \leq 1.75)) = 2 \cdot 0.04 = 0.08.$$

Since $0.08 > 0.05$ we fail to reject Kuzma's claim.

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We'll run our test in two ways. Next, let's compare our test statistic to $z_{1-\alpha/2}$.
Since $\alpha = 0.05$ we get that $z_{1-\alpha/2} = z_{0.975} = 1.96$. Because

$$|t^*| = 1.75 < 1.96 = z_{0.975}$$

we again fail to reject Kuzma's claim

Let's use these same estimates, $\hat{\lambda} = 19$ and $\hat{\sigma}_\lambda^2 = 16$, to construct a 95% confidence interval for the true parameter $\lambda = \beta_0 + 3\beta_1$.

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- Since we have assumed that the true relationship between Y (points per game) and X (number of mental health days taken per month) is linear then $\lambda = \mathbb{E}[Y|X = 3]$.

  - By linear we mean that $\mathbb{E}[Y|X = x] = \beta_0 + \beta_1 \cdot x$

  - Otherwise we can view $\lambda = \beta_0 + 3\beta_1$ as an approximation of $\mathbb{E}[Y|X = 3]$.

Let's use these same estimates, $\hat{\lambda} = 19$ and $\hat{\sigma}_\lambda^2 = 16$, to construct a 95% confidence interval for the true parameter $\lambda = \beta_0 + 3\beta_1$.

From above we have that a 95% confidence interval for λ can be constructed

$$\hat{\lambda} \pm z_{0.975} \frac{\hat{\sigma}_\lambda}{\sqrt{n}} = 19 \pm 1.96 \frac{4}{7}.$$

So that we are 95% confident that the true value of $\lambda = \beta_0 + 3\beta_1$ lies in the inteval $[17.88, 20.12]$.

- How could we use this interval to test the hypothesis $H_0 : \lambda = 20$ vs $H_1 : \lambda \neq 20$ at level $\alpha = 0.05$?

Inference: Linear Combinations of Parameters

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So that we are 95% confident that the true value of $\lambda = \beta_0 + 3\beta_1$ lies in the inteval $[17.88, 20.12]$.

- How could we use this interval to test the hypothesis $H_0 : \lambda = 20$ vs $H_1 : \lambda \neq 20$ at level $\alpha = 0.05$?

- What about testing this hypothesis at level $\alpha = 0.01$?

Questions?

Recall that in order to conduct inference and derive an asymptotic distribution for $\hat{\beta}_0$ and $\hat{\beta}_1$ we made the following assumptions about the underlying distribution of (Y, X):

- Random Sampling: Assume that $\{Y_i, X_i\}$ are independently and identically distributed; $(Y_i, X_i) \overset{\text{i.i.d}}{\sim} (Y, X)$

- Homoskedasticity: Assume that $\text{Var}(\epsilon \mid X = x) = \sigma_\epsilon^2$ for all possible values of x.

- Rank Condition: There must be at least two distinct values of X that appear in the population.

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- Rank Condition: There must be at least two distinct values of X that appear in the population.

In the last set of slides we mentioned that Homoskedasticity was a strong assumption that we will want to relax. Let's look more into why this is the case and how to relax the assumption.

Since our linear model relates Y and X via the following equation

$$Y = \beta_0 + \beta_1 X + \epsilon$$

the requirement that $\mathrm{Var}(\epsilon|X = x) = \sigma_\epsilon^2$ for all $x \in \mathrm{supp}(X)$ is implicitly requiring that $\mathrm{Var}(Y|X = x) = \mathrm{Var}(\epsilon|X = x) = \sigma_\epsilon^2$ for all $x \in \mathrm{supp}(X)$.

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- Intuitively this means that X has no information about the spread of Y.

- The variance of Y is the same for all values of X.

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To see why homoskedasticity is a restrictive assumption, let's turn to some examples.

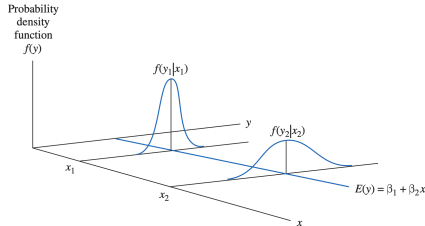- If homoskedasticity is violated we say that the errors ϵ exhibit heteroskedasticity

Example 1: Let Y be food expenditure and X be household income and suppose we want to estimate the linear model.

$$Y = \beta_0 + \beta_1 X + \epsilon.$$

Homoskedasticity requires that the variance of food expenditures is the same for all levels of income.
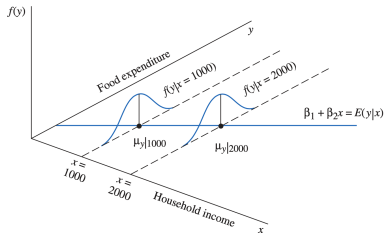
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Is this a reasonable assumption?

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Homoskedasticity requires that the variance of food expenditures is the same for all levels of income.

Is this a reasonable assumption?

- Low income individuals don't have much choice on how much to spend on food

Example 1: Let Y be food expenditure and X be household income and suppose we want to estimate the linear model.

$$Y = \beta_0 + \beta_1 X + \epsilon.$$

Homoskedasticity requires that the variance of food expenditures is the same for all levels of income.

Is this a reasonable assumption?

- Low income individuals don't have much choice on how much to spend on food

- High income individuals may have high variance reflecting variance in taste
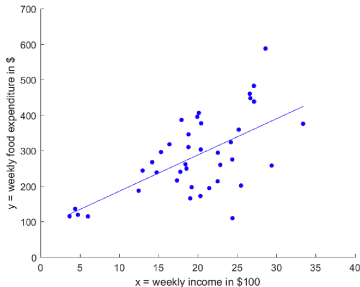
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Homoskedasticity requires that the variance of food expenditures is the same for all levels of income. Instead we may expect the spread of food expenditure to depend on income

Inference: Heteroskedasticity

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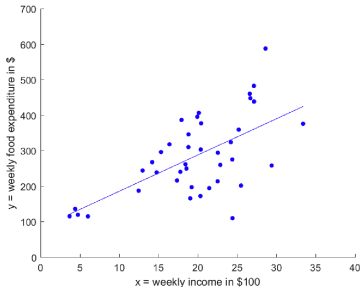
- Residuals look more spread out for higher income levels, suggests homoskedasticity is violated
- Not a formal test (using $\hat{\epsilon}$ instead of ϵ), but suggestive

Example 2: Let Y be wages and X be years of education and consider the model

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In this context homoskedasticity requires that the variance in wages is the same for all levels of education. Is this a reasonable assumption?

- High school graduates may not have access to as many career paths
- College graduates can range from English PhDs to engineers at Google

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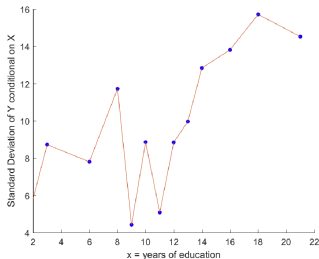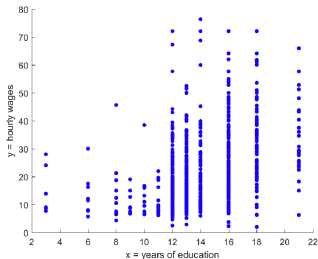
- Looking at the data we see that variance in wages increases increases after high school

Questions?

So, suppose we look at our data and suspect homoskedasticity is violated. What do we do now?

- Won't need to adjust our estimator

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$$\sqrt{n}\left(\hat{\beta}_1 - \beta_1\right) \approx \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\epsilon_i(X_i - \mu_X)}{\sigma_X^2}.$$

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We then applied the central limit theorem to $\frac{1}{\sqrt{n}}\sum_{i=1}^n \epsilon_i(X_i - \mu_X)$: approximately for large n,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n \epsilon_i(X_i - \mu_X) \sim N\left(0, \text{Var}(\epsilon(X - \mu_X))\right).$$

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- The only time we used homoskedasticity was to decompose $\text{Var}(\epsilon(X - \mu_X)) = \sigma_\epsilon^2 \sigma_X^2$. This simplified estimation but is not necessary.

Without heteroskedasticity we can use the same logic as before without decomposing $\operatorname{Var}(\epsilon(X - \mu_X)) = \sigma_\epsilon^2 \sigma_X^2$ to get

$$\sqrt{n}(\hat{\beta}_1 - \beta_1) \sim N\left(0, \frac{\operatorname{Var}(\epsilon(X - \mu_X))}{(\sigma_X^2)^2}\right) \implies \hat{\beta}_1 \sim N\left(\beta_1, \frac{\operatorname{Var}(\epsilon(X - \mu_X))}{n(\sigma_X^2)^2}\right).$$

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- Asymptotic variance is different now however, and will require a different estimator.

- Using the wrong variance renders our inference useless as we will not be accurately computing objects like

$$\Pr(|\hat{\beta}_1| > 5 | \beta_1 = 0).$$

As mentioned above, all that needs to be done to relax homoskedasticity is find a way to estimate the asymptotic variance of $\hat{\beta}_1$:

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  ○ Note that $\frac{1}{n} \sum_{i=1}^n \hat{\epsilon}_i X_i = \bar{X} \frac{1}{n} \sum_{i=1}^n \hat{\epsilon}_i = 0$ by the first order conditions for $\hat{\beta}_1$ and $\hat{\beta}_0$, respectively. So $\widehat{W_i} = \frac{1}{n} \sum_{i=1}^n \hat{W}_i = 0$.

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  - Since $\hat{\epsilon}_i \to \epsilon_i$ and $\bar{X} \to \mu_X$, $\hat{W}_i \to \epsilon_i(X - \mu_X)$ and so we have a consistent estimator for $\text{Var}(\epsilon(X - \mu_X))$ by law of large numbers.

To summarize under heteroskedasticity we have that (approximately for large n)

$$\sqrt{n}(\hat{\beta}_1 - \beta_1) \sim N(0, \sigma^2_{\beta_1}) \implies \hat{\beta}_1 \sim N(\beta_1, \sigma^2_{\beta_1}/n),$$

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- Once the heteroskedastic-consistent variances/standard errors/covariances are computed inference and confidence intervals are computed the same as before.

Questions?

Table of Contents

Let's switch tacks a bit and turn from inference to evaluating our model. Recall that we chose to model the relationship between Y and X by estimating the line of best fit between Y and X:

$$\beta_0, \beta_1 = \arg \min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E}\left[(Y - \tilde{\beta}_0 - \tilde{\beta}_1 X)^2\right].$$

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Now, however, let's consider a different question. Is this even a good model?

When we are answering this question we are essentially interested in how our model does as a tool to predict Y using X.

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This is a somewhat different goal than in inference when we are interested in interpreting the parameters β_0 and β_1 to learn about the underlying relationship between Y and X.

- Inference tends to be useful when thinking about policies to implement, want to know about average effects

Prediction can also be very useful though. Let's consider some examples.

Evaluating our Model: Prediction

Prediction can also be very useful though. Let's consider some examples.

Example: How can Amazon do two day delivery?

- In order to deliver an item within two days, an item has to be in stock in a warehouse nearby
- If too many people buy the item at once in a certain area the warehouse will run out of stock
- Amazon has to be able to accurately forecast/predict demand in certain areas.

Evaluating our Model: Prediction

Prediction can also be very useful though. Let's consider some examples.

Example: How much power should the energy grid generate?

- Energy grid must have enough supply to meet demand
- But it is costly and takes time to adjust production levels
- Energy suppliers must have good forecasts of demand

Prediction can also be very useful though. Let's consider some examples.

Example: How much should a pension fund keep in liquid funds?

- Fund must be able to pay all it's obligations
- Some portion of funds are invested and returns are random. People also retire/die at random times.
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Note that in these examples we are not per-se interested in interpreting the parameters of our linear regression model. We are just interested in using our linear regression model to predict Y using X.

Suppose we had no information on X. What is the best we can do in terms of predicting Y?

- If we just have information on income and we are given a random name, what should we predict their income to be?

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Since we have no additional information, we will make the same prediction for all new observations. Want to choose a value $a*$ that minimizes

$$a^* = \arg\min_a \mathbb{E}[(Y - a)^2].$$

That is a^* is "closest" to Y on average.

Taking first order conditions, we see that a^* solves

$$-2\mathbb{E}[(Y - a^*)] = 0 \implies a^* = \mathbb{E}[Y].$$

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The best predictor of Y with no additional information is just $\mathbb{E}[Y]$!

- This is intuitive enough

Now that we have information on X we have tried to use this information to predict Y by estimating a line of best fit (linear model) between Y and X:

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- Obviously cannot evaluate this directly since we don't know β_0, β_1 and $\mathbb{E}[Y]$

- Instead we will see how much closer $\hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_1 X_i$ is to Y_i on average than $\bar{Y}$.

Let's recall the following equalities implied by the first order conditions of our estimating equations

$$\hat{\beta}_0, \hat{\beta}_1 = \arg \min_{b_0, b_1} \frac{1}{n} \sum_{i=1}^{n} (Y_i - b_0 - b_1 X_i)^2.$$

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From the first order condition for $\hat{\beta}_0$:

$$\frac{1}{n} \sum_{i=1}^n (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) = \frac{1}{n} \sum_{i=1}^n \hat{\epsilon}_i = 0.$$

From the first order condition for $\hat{\beta}_1$:

$$\frac{1}{n} \sum_{i=1}^n (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) X_i = \frac{1}{n} \sum_{i=1}^n \hat{\epsilon}_i X_i = 0.$$

Recall that, by definition of $\hat{\epsilon}_i = Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i$

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and that after solving for $\hat{\beta}_0$ we get

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Using these we get that

$$Y_i - \bar{Y} = (\hat{Y}_i - \bar{Y}) + \hat{\epsilon}_i$$
$$\hat{Y}_i - \bar{Y} = \hat{\beta}_1 (X_i - \bar{X})$$

Now let's use these and decompose the sum:

$$\overbrace{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}^{\text{Total variance in } Y} = \sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2 + \frac{1}{n}\sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})\hat{\epsilon}_i + \frac{1}{n}\sum_{i=1}^{n}\hat{\epsilon}_i^2$$

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$$= \underbrace{\frac{1}{n}\sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2}_{\text{Explained by model with } X} + \underbrace{\frac{1}{n}\sum_{i=1}^{n}\hat{\epsilon}_i^2}_{\text{unexplained by model}}$$

Often we multiply both sides of the last equation by n and label the resulting components:

$$\sum_{i=1}^{n}(Y_i - \bar{Y})^2 = \sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2 + \sum_{i=1}^{n}\hat{\epsilon}_i^2.$$

1. $\sum_{i=1}^{n}(Y_i - \bar{Y})^2$: SST (Total Sum Of Squares)

   ○ Captures how much total variation there is in Y.

   ○ Can think of this as the sum of squared errors from just using $\bar{Y}$ to predict Y.

   ○ Note that this is just the sample variance of Y multiplied by n

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$$\sum_{i=1}^{n}(Y_i - \bar{Y})^2 = \sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2 + \sum_{i=1}^{n}\hat{\epsilon}_i^2.$$

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   - Can think of this as the sum of squared errors from just using $\bar{Y}$ to predict Y.
   - Note that this is just the sample variance of Y multiplied by n

2. $\sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2$: SSR (Sum of Squares due to Regression)
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   - This is the sample variance of $\hat{Y}_i$ multiplied by n

Often we multiply both sides of the last equation by n and label the resulting components:

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3. $\sum_{i=1}^{n}\hat{\epsilon}_i^2$: SSE (Sum of Squared Errors)
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Using the decomposition

$$\underbrace{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}_{\text{SST}} = \underbrace{\sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2}_{\text{SSR}} + \underbrace{\sum_{i=1}^{n}\hat{\epsilon}_i^2}_{\text{SSE}}.$$

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- R^2 is generally reported when running a regression by almost any statistical software.
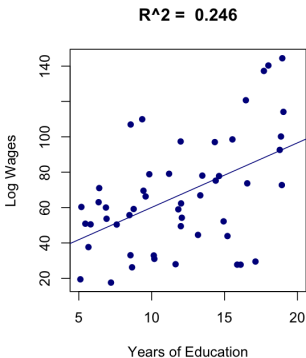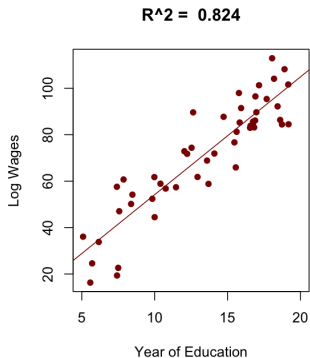
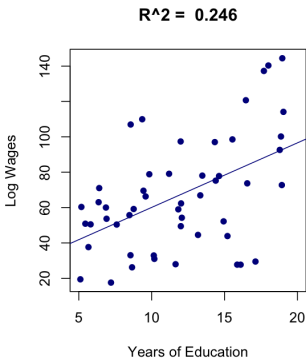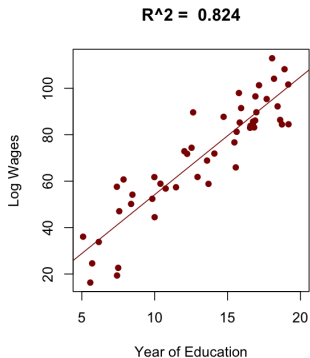Let's see how this looks with an example from two different datasets

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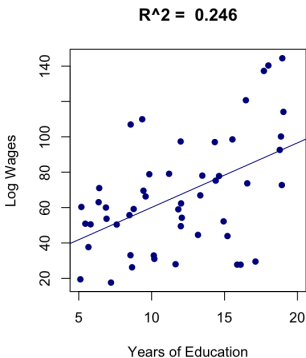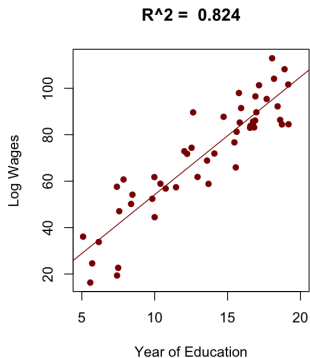
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Let's see how this looks with an example from two different datasets

- In which dataset does the regression line look closer to the data?
- Does homoskedasticity look to be violated here?

Table of Contents

Suppose we fit our linear model and find a low R^2. There are two main reasons this could be happening.

1. Knowing X simply does not give us much information about Y

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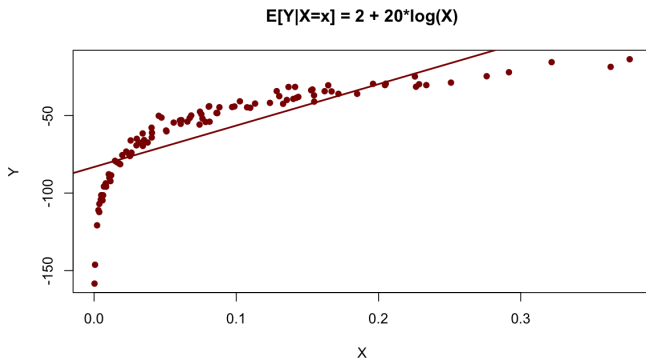2. The true relationship between X and Y is non-linear and we are trying to fit a linear model.

The first problem we can't do much to address, other than trying to collect more right hand side variables. The second problem however we can try and address by transforming our data.

Let's see an example of this. Suppose we collect our data and it looks like below

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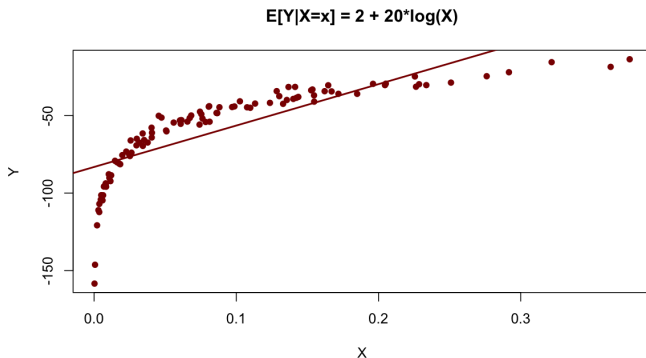
The relationship between X and Y is clearly non-linear, but we are trying to fit a line through it. This hurts our model preformance.

Other reasons that we may think that the relationship between Y and X is non-linear

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  - Suppose Y is amount spent on groceries and X is income.
  - We expect that as income rises people may start shopping at Whole Foods or spending more on nicer ingredients.
  - But this will probably level off for high levels of income. A linear model imposes that $\frac{d}{dx}\hat{Y}(x) = \frac{d}{dx}(\hat{\beta}_0 + \hat{\beta}_1 x) = \hat{\beta}_1$ for all x.

So, given that we believe the relationship between Y and X to be nonlinear, what can we do about this?

So, given that we believe the relationship between Y and X to be nonlinear, what can we do about this?

Before we move on to something fancier, let's try just transforming our data and running a linear regression:

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So, given that we believe the relationship between Y and X to be nonlinear, what can we do about this?

Before we move on to something fancier, let's try just transforming our data and running a linear regression:

$$f(Y) = \beta_0 + \beta_1 g(X) + \epsilon.$$

Common functions to be used here are $\ln(z)$, $\sqrt{z}$, and various polynomials; z^2, z^3, etc.

Modeling Choices: Transforming our Data

Let's return to our data from before and see how this would work.

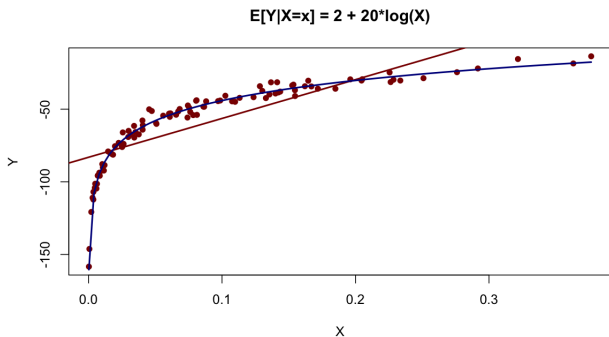

E[Y|X=x] = 2 + 20*log(X)
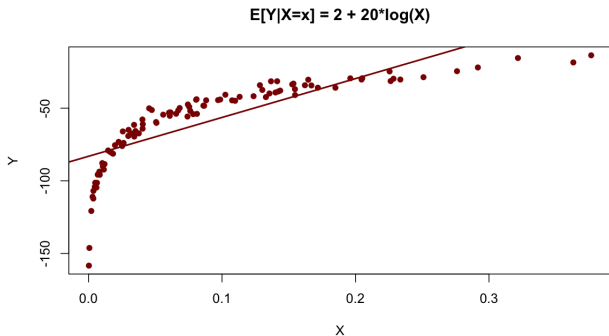
Instead of fitting the model $Y = \beta_0 + \beta_1 X + \epsilon$ let's try fitting the model
$Y = \beta_0 + \beta_1 \ln(X) + \epsilon$

- How is this done? Choose our estimates

$$\hat{\beta}_0, \hat{\beta}_1 = \arg\min_{b_0, b_1} \frac{1}{n} \sum_{i=1}^{n} \left(Y_i - b_0 - b_1 \cdot \ln(X_i) \right)^2 .$$

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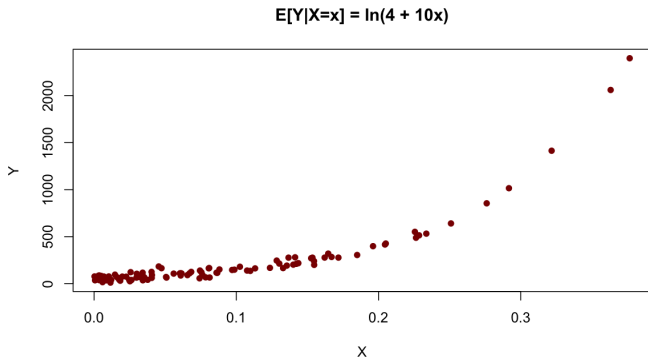

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Instead of fitting the model $Y = \beta_0 + \beta_1 X + \epsilon$ let's try fitting the model
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- We can see that the blue regression line is much closer to the data on average
  ($R^2 = 0.988$ vs $R^2 = 0.6536$)

Let's see another example. Suppose our data looks like the below.

E[Y|X=x] = ln(4 + 10x)

First, let's try fitting a non-transformed linear regression: $Y = \beta_0 + \beta_1 X + \epsilon$

Let's see another example. Suppose our data looks like the below.

E[Y|X=x] = ln(4 + 10x)

We see that this line does not appear to be fitting the data so well ($R^2 = 0.692$). Because Y apears to grow exponentially with X, we may want to try transforming Y.

Let's see another example. Suppose our data looks like the below.

E[Y|X=x] = ln(4 + 10x)

The blue curve is our estimate of the model $\ln(Y) = \beta_0 + \beta_1 X + \epsilon$. This fits the data much better and gives $R^2 = 0.8553$. We estimate the parameters via

$$\hat{\beta}_0, \hat{\beta}_1 = \arg\min_{b_0, b_1} \frac{1}{n} \sum_{i=1}^{n} (\ln(Y_i) - b_0 - b_1 X_i)^2.$$

In all of the above, notice that from an estimation and inference perspective, nothing much has changed. We can estimate our parameters and conduct inference just as before, but while treating our data as $\{f(Y_i), g(X_i)\}$ as opposed to $\{Y_i, X_i\}$.

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we find that, with $n = 49$, $\hat{\beta}_1 = 0.5$, $\sigma^2_{\ln(X)} = \frac{1}{n} \sum_{i=1}^{n} (\ln(X_i) - \overline{\ln(X)})^2 = 4$ and $\hat{\sigma}^2_\epsilon = \frac{1}{n} \sum_{i=1}^{n} (\ln(Y_i) - \hat{\beta}_0 - \hat{\beta}_1 \ln(X_i))^2 = 4$.

Assuming homoskedasticity, let's use this information to construct a 95% confidence interval for β_1.

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We use the same formula from before to calculate $\hat{\sigma}^2_{\beta_1} = \frac{\hat{\sigma}^2_\epsilon}{\hat{\sigma}^2_{\ln(X)}} = \frac{4}{4} = 1$.

Then, using $z_{0.975} = 1.96$, a 95% confidence interval is contructed

$$\hat{\beta}_1 \pm 1.96 \frac{\hat{\sigma}_{\beta_1}}{\sqrt{n}} = 0.5 \pm 1.96 \frac{1}{7} = [0.22, 0.78].$$

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  - Recall that we would reject $H_0 : \beta_1 = 0$ against $H_1 : \beta_1 \neq 0$ at level $\alpha = 0.5$.

Questions?

What changes, however, is our interpretation of our parameters. Before we interpreted:

- β_0: Expected value of Y when $X = 0$.

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In the model $f(Y) = \beta_0 + \beta_1 g(X) + \epsilon$, we have the following interpretations of β_0 and β_1.

- β_0: Expected value of $f(Y)$ when $g(X) = 0$

- $\beta_1 g'(x)$: Expected change in $f(Y)$ at $X = x$ when X increases by one unit.

Example: Suppose X represents the square footage of a house and Y represents is final sales price (in tens of thousands of dollars). We estimate the following model

$$Y = \beta_0 + \beta_1 X^2 + \epsilon$$

and find that $\hat{\beta}_0 = 50$ and $\hat{\beta}_1 = 2$.

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How do we interpret these parameter estimates?

- $\hat{\beta}_0 = 50$: The expected sales price of an empty lot ($X^2 = 0 \iff X = 0$) is $50,000.

- $\hat{\beta}_1 = 2$: Taking derivatives gives that $\frac{d}{dx}\hat{Y} = 2\hat{\beta}_1 X$. We expect that a one square foot increase in home size at square footage x will be associated with a $40,000·x increase in sales price.

A useful approximation that we use here is that a one unit increase in $\ln(Z)$ is about a 100% increase in Z and vice versa, a 1% increase in Z is associated with a $1/100$ unit increase in $\ln(Z)$.

This is useful for interpreting the parameters of various models.

- Suppose we model $Y = \beta_0 + \beta_1 \ln(X) + \epsilon \implies$ a 1% increase in X is associated with a $\beta_1/100$ unit change in Y.

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Modeling Choices: Linear Transformations

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$$\hat{\beta}_1^\circ = \frac{\frac{1}{n}\sum_{i=1}^n (\tilde{Y} - \bar{\tilde{Y}})(X_i - \bar{X})}{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})}$$

$$= c \underbrace{\frac{\frac{1}{n}\sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X})}{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2}}_{\text{estimated slope parameter from } Y \text{ on } X}$$

So we see that the slope parameter simply get's scaled up or down by c: $\hat{\beta}_1^\circ = c\hat{\beta}_1$.

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So we see that the slope parameter simply get's scaled up or down by c: $\hat{\beta}_1^\circ = c\hat{\beta}_1$.

The intercept parameter also gets scaled by c:

$$
\hat{\beta}_0^\circ = \bar{\tilde{Y}} - \hat{\beta}_1^\circ \bar{X} = c\bar{Y} - c\hat{\beta}_1 \bar{X} = c(\bar{Y} - \hat{\beta}_1 \bar{X}) = c\hat{\beta}_0.
$$

Now let's see what happens if we replace X with $\tilde{X} = cX$ for some $c \neq 0$. Let's consider estimating the model $Y = \beta_0^\circ + \beta_1^\circ \tilde{X} + \epsilon$.

$$\begin{aligned}
\hat{\beta}_1^\circ &= \frac{\frac{1}{n}\sum_{i=1}^n (Y_i - \bar{Y})(\tilde{X}_i - \bar{\tilde{X}})}{\frac{1}{n}\sum_{i=1}^n (\tilde{X}_i - \bar{\tilde{X}})^2} \\
&= \frac{c\frac{1}{n}\sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X})}{c^2 \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2} \\
&= \frac{1}{c}\hat{\beta}_1
\end{aligned}$$

Here the slope parameter gets scaled by $\frac{1}{c}$, $\hat{\beta}_1^\circ = \frac{1}{c}\hat{\beta}_1$.

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$$
\begin{aligned}
\hat{\beta}_1^\circ &= \frac{\frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})(\tilde{X}_i - \bar{\tilde{X}})}{\frac{1}{n} \sum_{i=1}^n (\tilde{X}_i - \bar{\tilde{X}})^2} \\
&= \frac{c\frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X})}{c^2 \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2} \\
&= \frac{1}{c} \hat{\beta}_1
\end{aligned}
$$

Here the slope parameter gets scaled by $\frac{1}{c}$, $\hat{\beta}_1^\circ = \frac{1}{c}\hat{\beta}_1$.

The intercept parameter in this case doesn't get scaled at all:

$$
\hat{\beta}_0^\circ = \bar{Y} - \hat{\beta}_1^\circ \bar{\tilde{X}} = \bar{Y} - \frac{1}{c}\hat{\beta}_1(c\bar{X}) = \beta_0.
$$

Now let's see what happens if we replace X with $\tilde{X} = cX$ for some $c \neq 0$. Let's consider estimating the model $Y = \beta_0^\circ + \beta_1^\circ \tilde{X} + \epsilon$.

$$\begin{aligned}
\hat{\beta}_1^\circ &= \frac{\frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})(\tilde{X}_i - \bar{\tilde{X}})}{\frac{1}{n} \sum_{i=1}^n (\tilde{X}_i - \bar{\tilde{X}})^2} \\
&= \frac{c \frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X})}{c^2 \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2} \\
&= \frac{1}{c} \hat{\beta}_1
\end{aligned}$$

Here the slope parameter gets scaled by $\frac{1}{c}$, $\hat{\beta}_1^\circ = \frac{1}{c} \hat{\beta}_1$.

The intercept parameter in this case doesn't get scaled at all:

$$\hat{\beta}_0^\circ = \bar{Y} - \hat{\beta}_1^\circ \bar{\tilde{X}} = \bar{Y} - \frac{1}{c} \hat{\beta}_1 (c\bar{X}) = \beta_0.$$

- Average value of Y when $X = 0$ is the same as the average value of Y when $cX = 0$, they are the same event.

Questions?